The Finite Horizon Optimal Multi-Modes Switching Problem: the Viscosity Solution Approach.

Brahim EL-ASRI * and Said HAMADENE †

Abstract

In this paper we show existence and uniqueness of a solution for a system of m variational partial differential inequalities with inter-connected obstacles. This system is the deterministic version of the Verification Theorem of the Markovian optimal m-states switching problem. The switching cost functions are arbitrary. This problem is in relation with the valuation of firms in a financial market.

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1 Introduction

In this work we are concerned with the following system of m variational partial differential inequalities with inter-connected obstacles:

$$\begin{cases} \min\{v_i(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}(-g_{ij}(t,x)+v_j(t,x)), \\ \qquad\qquad -\partial_t v_i(t,x) - \mathcal{A}v_i(t,x) - \psi_i(t,x)\} = 0, \ \forall (t,x)\in[0,T]\times I\!\!R^k,\ i\in\mathcal{I}=\{1,...,m\}, \\ v_i(T,x)=0. \end{cases} \tag{1.1}$$

where g_{ij}, ψ_i are continuous functions, $\mathcal{A}$ an infinitesimal generator associated with a diffusion process and finally $\mathcal{I}^{-i} := \{1,...,i-1,i+1,...,m\}$.

This system is the deterministic version of the Verification Theorem of the optimal multi-modes switching problem in finite horizon. This problem, of real option type, can be introduced with the help of the following example:

*Université du Maine, Dépt. de Mathématiques, Equipe Stat. et Processus, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France. e-mails: brahim.el_Asri.etu@univ-lemans.fr

†Université du Maine, Dépt. de Mathématiques, Equipe Stat. et Processus, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France. e-mail: hamadene@univ-lemans.fr

Assume we have a power station/plant which produces electricity and which has several modes of production, e.g., the lower, the middle and the intensive modes. The price of electricity in the market, given by an adapted stochastic process $(X_t)_{t\leq T}$, fluctuates in reaction to many factors such as demand level, weather conditions, unexpected outages and so on. On the other hand, electricity is non-storable, once produced it should be almost immediately consumed. Therefore, as a consequence, the station produces electricity in its instantaneous most profitable mode known that when the plant is in mode $i \in \mathcal{I}$, the yield per unit time is given by means of ψ_i and, on the other hand, switching the plant from the mode i to the mode j is not free and generates expenditures given by g_{ij} and possibly by other factors in the energy market. So the manager of the power plant faces two main issues:

(i) when should her decide to switch the production from its current mode to another one?

(ii) to which mode the production has to be switched when the decision of switching is made?

In other words she faces the issue of finding the optimal strategy of management of the plant. This issue is in relation with the price of the power plant in the energy market.

For decades, optimal switching problems have attracted a lot of research activity (see e.g. [1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 17, 18, 19, 20, 22, 23, 21, 25, 26, 27, 28, 29] and the references therein). Especially in connection with valuation of firms and, questions related to the structural profitability of investment project or an industry whose production depends on the fluctuating market price of a number of underlying commodities or assets, ... Several variants of the problem we deal with here have been considered. In order to tackle those problems, authors use mainly two approaches. Either a probabilistic one ([10, 11, 17, 19, 22, 23]) or an approach which uses partial differential inequalities (PDIs for short) ([1, 2, 5, 13, 20, 29, 26]).

The PDIs approach turns out to study and to solve, in some sense, the system of m PDIs with inter-connected obstacles (1.1). Amongst the papers which consider the same problem as ours, and in the framework of viscosity solutions approach, the most elaborated works are certainly the ones by Tang and Yong [26], on the one hand, and by Djehiche et al. [11], on the other hand. In [26], the authors show existence and uniqueness of a solution for (1.1). Nevertheless the paper suffers from two facts: (i) the growth exponent at infinity of the functions ψ_i should be smaller that 2 ; (ii) the switching cost functions g_{ij} should not depend on x. The first issue of [26] has been treated by Djehiche et al.[11] since in their paper the authors show existence of the solution for 1.1 in the case when the growth of the functions ψ_i is of arbitrary polynomial type. The second issue of [26], i.e. considering the case when g_{ij} depending also on x, was right now, according to our knowledge, an open problem. Note that in [11], the question of uniqueness is not addressed. Therefore the main objective of our work, and this is the novelty of the paper, is to show existence and uniqueness of a solution in viscosity sense for the system when the functions ψ_i and g_{ij} are continuous depending also on x and satisfy an arbitrary polynomial

growth condition. We show also that the solution is unique in the class of continuous functions with polynomial growth.

This paper is organized as follows:

In Section 2, we formulate the problem and we give the related definitions. In Section 3, we introduce the optimal switching problem under consideration and give its probabilistic Verification Theorem. It is expressed by means of a Snell envelope of processes. Then we introduce the approximating scheme which enables to construct a solution for the Verification Theorem. Moreover we give some properties of that solution, especially the dynamic programming principle. Section 4 is devoted to the connection between the optimal switching problem, the Verification Theorem and the system of PDIs (1.1). This connection is made through backward stochastic differential equations with one reflecting obstacle in the case when randomness comes from a solution of a standard stochastic differential equation. Further we provide some estimate for the optimal strategy of the switching problem which in combination with the dynamic programming principle plays a crucial role. Finally we show that system (1.1) has a solution. In Section 5, we show that the solution of (1.1) is unique in the class of continuous functions which satisfy a polynomial growth condition. $\Box$

2 Assumptions and formulation of the problem

Throughout this paper T (resp. k) is a fixed real (resp. integer) positive constant. Let us now consider the followings:

(i) $b : [0,T] \times I\!R^k \to I\!R^k$ and $\sigma : [0,T] \times I\!R^k \to I\!R^{k\times d}$ are two continuous functions for which there exists a constant $C \geq 0$ such that for any $t \in [0,T]$ and $x, x' \in I\!R^k$

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|) \quad \text{and} \quad |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x-x'| \tag{2.1}$$

(ii) for $i, j \in \mathcal{I} = \{1, ..., m\}$, $g_{ij} : [0,T] \times I\!R^k \to I\!R$ and $\psi_i : [0,T] \times I\!R^k \to I\!R$ are continuous functions and of polynomial growth, *i.e.*, there exist some positive constants C and γ such that for each $i, j \in \mathcal{I}$:

$$|\psi_i(t,x)| + |g_{ij}(t,x)| \leq C(1+|x|^\gamma), \; \forall (t,x) \in [0,T] \times I\!R^k. \tag{2.2}$$

Moreover we assume that there exists a constant $\alpha > 0$ such that for any $(t,x) \in [0,T] \times I\!R^k$,

$$min\{g_{ij}(t,x), i, j \in \mathcal{I}, \quad i \neq j\} \geq \alpha. \tag{2.3}$$

We now consider the following system of m variational inequalities with inter-connected obstacles:

$\forall\ i \in \mathcal{I}$

$$\begin{cases} \min\Big\{v_i(t,x) - \max\limits_{j\in\mathcal{I}^{-i}}(-g_{ij}(t,x)+v_j(t,x)), -\partial_t v_i(t,x) - \mathcal{A}v_i(t,x) - \psi_i(t,x)\Big\} = 0, \\ v_i(T,x) = 0, \end{cases} \tag{2.4}$$

where $\mathcal{I}^{-i} := \mathcal{I} - \{i\}$ and $\mathcal{A}$ is the following infinitesimal generator:

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1,k}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1,k} b_i(t,x)\frac{\partial}{\partial x_i}\,; \tag{2.5}$$

hereafter the superscript $(^*)$ stands for the transpose, Tr is the trace operator and finally $< x, y >$ is the inner product of $x, y \in I\!R^k$.

The main objective of this paper is to focus on the existence and uniqueness of the solution in viscosity sense of (2.4) whose definition is:

Definition 1 *Let $(v_1, ..., v_m)$ be a m-uplet of continuous functions defined on $[0,T]\times I\!R^k$, $I\!R$-valued and such that such that $v_i(T,x) = 0$ for any $x \in I\!R^k$ and $i \in \mathcal{I}$. The m-uplet $(v_1, ..., v_m)$ is called:*

(i) a viscosity supersolution (resp. subsolution) of the system (2.4) if for each fixed $i \in \mathcal{I}$, for any $(t_0, x_0) \in [0,T]\times I\!R^k$ and any function $\varphi_i \in C^{1,2}([0,T]\times I\!R^k)$ such that $\varphi_i(t_0,x_0) = v_i(t_0,x_0)$ and (t_0,x_0) is a local maximum of $\varphi_i - v_i$ (resp. minimum), we have:

$$\begin{aligned} \min\Big\{ & v_i(t_0,x_0) - \max_{j\in\mathcal{I}^{-i}}(-g_{ij}(t_0,x_0)+v_j(t_0,x_0)), \\ & -\partial_t\varphi_i(t_0,x_0) - \mathcal{A}\varphi_i(t_0,x_0) - \psi_i(t_0,x_0)\} \geq 0 \quad (\textit{resp.} \leq 0). \end{aligned} \tag{2.6}$$

(ii) a viscosity solution if it is both a viscosity supersolution and subsolution. □

There is an equivalent formulation of this definition (see e.g. [6]) which we give because it will be useful later. So firstly we define the notions of superjet and subjet of a continuous function v.

Definition 2 *Let $v \in C((0,T)\times I\!R^k)$, (t,x) an element of $(0,T)\times I\!R^k$ and finally S_k the set of $k\times k$ symmetric matrices. We denote by $J^{2,+}v(t,x)$ (resp. $J^{2,-}v(t,x)$), the superjets (resp. the subjets) of v at (t,x), the set of triples $(p,q,X) \in I\!R \times I\!R^k \times S_k$ such that:*

$$\begin{gathered} v(s,y) \leq v(t,x) + p(s-t) + \langle q, y-x\rangle + \tfrac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2) \\ (\textit{resp.} \quad v(s,y) \geq v(t,x) + p(s-t) + \langle q, y-x\rangle + \tfrac{1}{2}\langle X(y-x), y-x\rangle + o(|s-t| + |y-x|^2)).\square \end{gathered}$$

Note that if $\varphi - v$ has a local maximum (resp. minimum) at (t,x), then we obviously have:

$$\big(D_t\varphi(t,x), D_x\varphi(t,x), D^2_{xx}\varphi(t,x)\big) \in J^{2,-}v(t,x) \ \ (\text{resp. } J^{2,+}v(t,x)).\square$$

We now give an equivalent definition of a viscosity solution of the parabolic system with interconnected obstacles (2.4).

Definition 3 *Let $(v_1, ..., v_m)$ be a m-uplet of continuous functions defined on $[0,T] \times I\!R^k$, $I\!R$-valued and such that $(v_1, ..., v_m)(T,x) = 0$ for any $x \in I\!R^k$. The m-uplet $(v_1, ..., v_m)$ is called a viscosity supersolution (resp. subsolution) of (2.4) if for any $i \in \mathcal{I}$, $(t,x) \in (0,T) \times I\!R^k$ and $(p,q,X) \in J^{2,-}v_i(t,x)$ (resp. $J^{2,+}v_i(t,x)$),*

$$min\left\{v_i(t,x) - \max_{j\in\mathcal{I}^{-i}}(-g_{ij}(t,x) + v_j(t,x)), -p - \frac{1}{2}Tr[\sigma^* X\sigma] - \langle b, q\rangle - \psi_i(t,x)\right\} \geq 0 \; (resp. \leq 0).$$

It is called a viscosity solution it is both a viscosity subsolution and supersolution .□

As pointed out previously we will show that system (2.4) has a unique solution in viscosity sense. This system is the deterministic version of the optimal m-states switching problem which is well documented in [11] and which we will describe briefly in the next section.

3 The optimal m-states switching problem

3.1 Setting of the problem

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the P-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions, *i.e.*, it is right continuous and complete. Furthermore, let:

- $\mathcal{P}$ be the σ-algebra on $[0,T] \times \Omega$ of $\mathbf{F}$-progressively measurable sets;

- $\mathcal{M}^{2,k}$ be the set of $\mathcal{P}$-measurable and $I\!R^k$-valued processes $w = (w_t)_{t\leq T}$ such that $E[\int_0^T |w_s|^2 ds] < \infty$ and $\mathcal{S}^2$ be the set of $\mathcal{P}$-measurable, continuous processes $w = (w_t)_{t\leq T}$ such that $E[\sup_{t\leq T} |w_t|^2] < \infty$;

- for any stopping time $\tau \in [0,T]$, $\mathcal{T}_\tau$ denotes the set of all stopping times θ such that $\tau \leq \theta \leq T$.

□

The problem of multiple switching can be described through an example as follows. Assume we have a plant which produces a commodity, *e.g.* a power station which produces electricity. Let $\mathcal{I}$ be the set of all possible activity modes of the production of the commodity. A management strategy of the plant consists, on the one hand, of the choice of a sequence of nondecreasing stopping times $(\tau_n)_{n\geq 1}$ ($i.e. \tau_n \leq \tau_{n+1}$ and $\tau_0 = 0$) where the manager decides to switch the activity from its current mode to another one. On the other hand, it consists of the choice of the mode ξ_n, a r.v. $\mathcal{F}_{\tau_n}$-measurable with values in $\mathcal{I}$, to which the production is switched at τ_n from its current mode. Therefore the admissible management strategies of the plant are the pairs $(\delta, \xi) := ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ and the set of these strategies is denoted by $\mathcal{D}$.

Let now $X := (X_t)_{0\leq t\leq T}$ be an $\mathcal{P}$-measurable, $I\!R^k$-valued continuous stochastic process which stands for the market price of k factors which determine the market price of the commodity. On the other

hand, assuming that the production activity is in mode 1 at the initial time $t = 0$, let $(u_t)_{t \le T}$ denote the indicator of the production activity's mode at time $t \in [0, T]$:

$$u_t = 1\!\!1_{[0,\tau_1]}(t) + \sum_{n \ge 1} \xi_n 1\!\!1_{(\tau_n,\tau_{n+1}]}(t). \tag{3.1}$$

Then for any $t \le T$, the state of the whole economic system related to the project at time t is represented by the vector :

$$(t, X_t, u_t) \in [0, T] \times I\!\!R^k \times \mathcal{I}. \tag{3.2}$$

Finally, let $\psi_i(t, X_t)$ be the instantaneous profit when the system is in state (t, X_t, i), and for $i, j \in \mathcal{I} \quad i \ne j$, let $g_{ij}(t, X_t)$ denote the switching cost of the production at time t from current mode i to another mode j. Then if the plant is run under the strategy $(\delta, \xi) = ((\tau_n)_{n\ge1}, (\xi_n)_{n\ge1})$ the expected total profit is given by:

$$J(\delta, \xi) = E[\int_0^T \psi_{u_s}(s, X_s)ds - \sum_{n\ge1} g_{u_{\tau_{n-1}} u_{\tau_n}}(\tau_n, X_{\tau_n}) 1\!\!1_{[\tau_n<T]}].$$

Therefore the problem we are interested in is to find an optimal strategy, *i.e*, a strategy (δ^*, ξ^*) such that $J(\delta^*, \xi^*) \ge J(\delta, \xi)$ for any $(\delta, \xi) \in \mathcal{D}$.

Note that in order that the quantity $J(\delta, \xi)$ makes sense we assume throughout this paper that for any $i, j \in \mathcal{I}$ the processes $(\psi_i(t, X_t))_{t\le T}$ and $(g_{ij}(t, X_t))_{t\le T}$ belong to $\mathcal{M}^{2,1}$ and $\mathcal{S}^2$ respectively. On the other hand there is a bijective correspondence between the pairs (δ, ξ) and the pairs (δ, u). Therefore throughout this paper one refers indifferently to (δ, ξ) or (δ, u).

3.2 The Verification Theorem

To tackle the problem described above Djehiche et al. [11] have introduced a Verification Theorem which is expressed by means of Snell envelope of processes. The Snell envelope of a stochastic process $(\eta_t)_{t\le T}$ of $\mathcal{S}^2$ (with a possible positive jump at T) is the lowest supermartingale $R(\eta) := (R(\eta)_t)_{t\le T}$ of $\mathcal{S}^2$ such that for any $t \le T$, $R(\eta)_t \ge \eta_t$. It has the following expression:

$$\forall t \le T, R(\eta)_t = esssup_{\tau\in\mathcal{T}_t} E[\eta_\tau|\mathbf{F}_t] \text{ and satisfies } R(\eta)_T = \eta_T.$$

For more details on the Snell envelope notion on can see e.g. [4, 14, 16].

The Verification Theorem for the m-states optimal switching problem is the following:

Theorem 1 *([11], Th.1) Assume that there exist m processes $(Y^i := (Y^i_t)_{0\le t\le T}, i = 1, ..., m)$ of $\mathcal{S}^2$ such that:*

$$\forall t \le T,\ Y^i_t = ess\ sup_{\tau\ge t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{j\in\mathcal{I}^{-i}}(-g_{ij}(\tau, X_\tau) + Y^j_\tau)1_{[\tau<T]}|\mathcal{F}_t], \quad Y^i_T = 0. \tag{3.3}$$

Then:

(i) $Y_0^1 = \sup_{(\delta,\xi)\in\mathcal{D}} J(\delta,u)$.

(ii) Define the sequence of $\mathbf{F}$*-stopping times* $\delta^* = (\tau_n^*)_{n\geq 1}$ *as follows :*

$$\begin{aligned}
\tau_1^* &= \inf\{s\geq 0, \quad Y_s^1 = \max_{j\in\mathcal{I}^{-1}}(-g_{1j}(s,X_s)+Y_s^j)\}\wedge T,\\
\tau_n^* &= \inf\{s\geq \tau_{n-1}^*, \quad Y_s^{u_{\tau_{n-1}^*}} = \max_{k\in\mathcal{I}\setminus\{u_{\tau_{n-1}^*}\}}(-g_{u_{\tau_{n-1}^*}k}(s,X_s)+Y_s^k)\}\wedge T, \quad \text{for} \quad n\geq 2,
\end{aligned}$$

where:

- $u_{\tau_1^*} = \sum_{j\in\mathcal{I}} j 1\!\!1_{\{\max_{k\in\mathcal{I}^{-1}}(-g_{1k}(\tau_1^*,X_{\tau_1^*})+Y_{\tau_1^*}^k) = -g_{1j}(\tau_1^*,X_{\tau_1^*})+Y_{\tau_1^*}^j\}}$;
- *for any* $n\geq 1$ *and* $t\geq \tau_n^*$, $Y_t^{u_{\tau_n^*}} = \sum_{j\in\mathcal{I}} 1\!\!1_{[u_{\tau_n^*}=j]} Y_t^j$
- *for any* $n\geq 2$, $u_{\tau_n^*} = l$ *on the set*

$$\left\{\max_{k\in\mathcal{I}\setminus\{u_{\tau_{n-1}^*}\}}(-g_{u_{\tau_{n-1}^*}k}(\tau_n^*,X_{\tau_n^*})+Y_{\tau_n^*}^k) = -g_{u_{\tau_{n-1}^*}l}(\tau_n^*,X_{\tau_n^*})+Y_{\tau_n^*}^l\right\}$$

with $g_{u_{\tau_{n-1}^*}k}(\tau_n^*,X_{\tau_n^*}) = \sum_{j\in\mathcal{I}} 1\!\!1_{[u_{\tau_{n-1}^*}=j]} g_{jk}(\tau_n^*,X_{\tau_n^*})$ *and* $\mathcal{I}\setminus\{u_{\tau_{n-1}^*}\} = \sum_{j\in\mathcal{I}} 1\!\!1_{[u_{\tau_{n-1}^*}=j]}\mathcal{I}^{-j}$.

Then the strategy (δ^*,u^*) *is optimal i.e.* $J(\delta^*,u^*)\geq J(\delta,u)$ *for any* $(\delta,u)\in\mathcal{D}$. □

The issue of existence of the processes $Y^1,...,Y^m$ which satisfy (3.3) is also addressed in [11]. Also for $n\geq 0$ let us define the processes $(Y^{1,n},...,Y^{m,n})$ recursively as follows: for $i\in\mathcal{I}$ we set,

$$Y_t^{i,0} = \text{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_i(s,X_s)ds|\mathcal{F}_t], \ 0\leq t\leq T, \tag{3.4}$$

and for $n\geq 1$,

$$Y_t^{i,n} = \text{ess sup}_{\tau\geq t} E[\int_t^\tau \psi_i(s,X_s)ds + \max_{k\in\mathcal{I}^{-i}}(-g_{ik}(\tau,X_\tau)+Y_\tau^{k,n-1})1\!\!1_{[\tau<T]}|\mathcal{F}_t], \ 0\leq t\leq T. \tag{3.5}$$

Then the sequence of processes $((Y^{1,n},...,Y^{m,n}))_{n\geq 0}$ have the following properties:

Proposition 1 *([11], Pro.3 and Th.2)*

(i) for any $i\in\mathcal{I}$ *and* $n\geq 0$*, the processes* $Y^{1,n},...,Y^{m,n}$ *are well-posed, continuous and belong to* $\mathcal{S}^2$*, and verify*

$$\forall t\leq T, \ Y_t^{i,n}\leq Y_t^{i,n+1}\leq E[\int_t^T\{\max_{i=1,m}|\psi_i(s,X_s)|\}ds|\mathcal{F}_t]; \tag{3.6}$$

(ii) there exist m *processes* $Y^1,...,Y^m$ *of* $\mathcal{S}^2$ *such that for any* $i\in\mathcal{I}$*:*

(a) $\forall t\leq T$, $Y_t^i = \lim_{n\to\infty}\nearrow Y_t^{i,n}$ *and*

$$E[\sup_{s\leq T}|Y_s^{i,n}-Y_s^i|^2]\to 0 \quad \text{as} \quad n\to+\infty$$

(b) $\forall t \leq T$,

$$Y_t^i = ess\ sup_{\tau \geq t} E[\int_t^\tau \psi_i(s, X_s)ds + \max_{k \in \mathcal{I}^{-i}}(-g_{ik}(\tau, X_\tau) + Y_\tau^k)\mathbb{1}_{[\tau<T]}|\mathcal{F}_t] \tag{3.7}$$

i.e. $Y^1, ..., Y^m$ *satisfy the Verification Theorem 1 ;*

(c) $\forall t \leq T$,

$$Y_t^i = esssup_{(\delta,\xi)\in\mathcal{D}_t^i} E[\int_t^T \psi_{u_s}(s, X_s)ds - \sum_{n\geq 1} g_{u_{\tau_{n-1}}u_{\tau_n}}(\tau_n, X_{\tau_n})\mathbb{1}_{[\tau_n<T]}]|\mathcal{F}_t] \tag{3.8}$$

where $\mathcal{D}_t^i = \{(\delta,\xi) = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ *such that* $u_0 = i$ *and* $\tau_1 \geq t\}$. *This characterization means that if at time* t *the production activity is in its regime* i *then the optimal expected profit is* Y_t^i.

(d) *the processes* $Y^1, ..., Y^m$ *verify the dynamical programming principle of the* m*-states optimal switching problem, i.e.,* $\forall t \leq T$,

$$Y_t^i = ess\ sup_{(\delta,u)\in\mathcal{D}_t^i} E[\int_t^{\tau_n} \psi_{u_s}(s, X_s)ds - \sum_{1\leq k\leq n} g_{u_{\tau_{k-1}}u_{\tau_k}}(\tau_k, X_{\tau_k})\mathbb{1}_{[\tau_k<T]} + \mathbb{1}_{[\tau_n<T]}Y_{\tau_n}^{u_{\tau_n}}|\mathcal{F}_t].\Box \tag{3.9}$$

Note that except $(ii-d)$, the proofs of the other points are given in [11]. The proof of $(ii-d)$ can be easily deduced in using relation (3.7). Actually from (3.7) for any $i \in \mathcal{I}$, $t \in [0,T]$ and $(\delta,\xi) \in \mathcal{D}_t^i$ we have:

$$Y_t^i \geq E[\int_t^{\tau_n} \psi_{u_s}(s, X_s)ds - \sum_{1\leq k\leq n} g_{u_{\tau_{k-1}}u_{u_{\tau_k}}}(\tau_k, X_{\tau_k})\mathbb{1}_{[\tau_k<T]} + \mathbb{1}_{[\tau_n<T]}Y_{\tau_n}^{u_{\tau_n}}|\mathcal{F}_t]. \tag{3.10}$$

Next using the optimal strategy we obtain the equality instead of inequality in (3.10). Therefore the relation (3.9) holds true. $\Box$

Remark 1 *Note that the characterization (3.8) implies that the processes* $Y^1, ..., Y^m$ *of* $\mathcal{S}^2$ *which satisfy the Verification Theorem are unique.*

4 Existence of a solution for the system of variational inequalities

4.1 Connection with BSDEs with one reflecting barrier

Let $(t,x) \in [0,T] \times I\!R^k$ and let $(X_s^{tx})_{s\leq T}$ be the solution of the following standard SDE:

$$dX_s^{tx} = b(s, X_s^{tx})ds + \sigma(s, X_s^{tx})dB_s \text{ for } t \leq s \leq T \text{ and } X_s^{tx} = x \text{ for } s \leq t \tag{4.1}$$

where the functions b and σ are the ones of (2.1). These properties of σ and b imply in particular that the process $(X_s^{tx})_{0\leq s\leq T}$ solution of the standard SDE (4.1) exists and is unique, for any $t \in [0,T]$ and $x \in I\!R^k$.

The operator $\mathcal{A}$ that is appearing in (2.5) is the infinitesimal generator associated with $X^{t,x}$. In the following result we collect some properties of $X^{t,x}$.

Proposition 2 *(see e.g. [24]) The process X^{tx} satisfies the following estimates:*

(i) For any $q \geq 2$, there exists a constant C such that

$$E[\sup_{0\leq s\leq T} |X_s^{tx}|^q] \leq C(1+|x|^q). \tag{4.2}$$

(ii) There exists a constant C such that for any $t, t' \in [0,T]$ and $x, x' \in I\!R^k$,

$$E[\sup_{0\leq s\leq T} |X_s^{tx} - X_s^{t'x'}|^2] \leq C(1+|x|^2)(|x-x'|^2 + |t-t'|).\Box \tag{4.3}$$

We are going now to introduce the notion of a BSDE with one reflecting barrier introduced in [15]. This notion will allow us to make the connection between the variational inequalities system (2.4) and the m-states optimal switching problem described in the previous section.

So let us introduce the deterministic functions $f : [0,T] \times I\!R^{k+1+d} \to I\!R$, $h : [0,T] \times I\!R^k \to I\!R$ and $g : I\!R^k \to I\!R$ continuous, of polynomial growth and such that $h(x,T) \leq g(x)$. Moreover we assume that for any $(t,x) \in [0,T] \times I\!R^k$, the mapping $(y,z) \in I\!R^{1+d} \mapsto f(t,x,y,z)$ is uniformly Lipschitz. Then we have the following result related to BSDEs with one reflecting barrier:

Theorem 2 *([15], Th. 5.2 and 8.5) For any $(t,x) \in [0,T] \times I\!R^k$, there exits a unique triple of processes $(Y^{t,x}, Z^{t,x}, K^{t,x})$ such that:*

$$\begin{cases} Y^{tx}, K^{tx} \in \mathcal{S}^2 \text{ and } Z^{tx} \in \mathcal{M}^{2,d};\ K^{tx} \text{ is non-decreasing and } K_0^{tx} = 0, \\ Y_s^{tx} = g(X_T^{tx}) + \int_s^T f(r, X_r^{tx}, Y_r^{tx}, Z_r^{tx})dr - \int_s^T Z_r^{tx} dB_r + K_T^{tx} - K_s^{tx},\ s \leq T \\ Y_s^{tx} \geq h(s, X_s^{tx}),\ \forall s \leq T \text{ and } \int_0^T (Y_r^{tx} - h(r, X_r^{tx}))dK_r^{tx} = 0. \end{cases} \tag{4.4}$$

Moreover the following characterization of $Y^{t,x}$ as a Snell envelope holds true:

$$\forall s \leq T,\ Y_s^{t,x} = esssup_{\tau\in\mathcal{T}_t} E[\int_t^\tau f(r, X_r^{tx}, Y_r^{tx}, Z_r^{tx})dr + h(\tau, X_\tau^{tx})1\!\!1_{[\tau<T]} + g(X_T^{tx})1\!\!1_{[\tau=T]}|\mathcal{F}_s]. \tag{4.5}$$

On the other hand there exists a deterministic continuous with polynomial growth function $u : [0,T] \times I\!R^k \to I\!R$ such that:

$$\forall s \in [t,T], Y_s^{t,x} = u(s, X_s^{t,x}).$$

Moreover the function u is the unique viscosity solution in the class of continuous function with polynomial growth of the following PDE with obstacle:

$$\begin{cases} \min\{u(t,x) - h(t,x), -\partial_t u(t,x) - \mathcal{A}u(t,x) - f(t,x,u(t,x), \sigma(t,x)^* \nabla u(t,x))\} = 0, \\ u(T,x) = g(x).\Box \end{cases}$$

4.2 Existence of a solution for the system of variational inequalities

Let $(Y_s^{1,tx}, ..., Y_s^{m,tx})_{0\le s\le T}$ be the processes which satisfy the Verification Theorem 1 in the case when the process $X \equiv X^{t,x}$. Therefore using the characterization (4.5), there exist processes $K^{i,tx}$ and $Z^{i,tx}$, $i \in \mathcal{I}$, such that the triples $(Y^{i,tx}, Z^{i,tx}, K^{i,tx})$ are unique solutions (thanks to Remark 1) of the following reflected BSDEs: for any $i = 1, ..., m$ we have,

$$\begin{cases} Y^{i,tx}, K^{i,tx} \in \mathcal{S}^2 \text{ and } Z^{i,tx} \in \mathcal{M}^{2,d};\ K^{i,tx} \text{ is non-decreasing and } K_0^{i,tx} = 0, \\ Y_s^{i,tx} = \int_s^T \psi_i(r, X_r^{tx})du - \int_s^T Z_r^{i,tx} dB_r + K_T^{i,tx} - K_s^{i,tx},\ \ 0 \le s \le T,\ Y_T^{i,tx} = 0, \\ Y_s^{i,tx} \ge \max\limits_{j\in\mathcal{I}^{-i}} (-g_{ij}(s, X_s^{tx}) + Y_s^{j,tx}),\ \ 0 \le s \le T, \\ \int_0^T (Y_r^{i,tx} - \max\limits_{j\in\mathcal{I}^{-i}} (-g_{ij}(r, X_r^{tx}) + Y_r^{j,tx}))dK_r^{i,tx} = 0. \end{cases} \tag{4.6}$$

Moreover we have the following result.

Proposition 3 *There are deterministic functions $v^1, ..., v^m : [0,T] \times I\!R^k \to I\!R$ such that:*

$$\forall (t,x) \in [0,T] \times I\!R^k, \forall s \in [t,T], Y_s^{i,tx} = v^i(s, X_s^{tx}),\ \ i = 1, ..., m.$$

Moreover the functions v^i, $i = 1, ..., m$, are lower semi-continuous and of polynomial growth.

Proof: For $n \ge 0$ let $(Y_s^{n,1,tx}, ..., Y_s^{n,m,tx})_{0\le s\le T}$ be the processes constructed in (3.4)-(3.5). Therefore using an induction argument and Theorem 2 there exist deterministic continuous with polynomial growth functions $v^{i,n}$ $(i = 1, ..., m)$ such that for any $(t,x) \in [0,T] \times I\!R^k$, $\forall s \in [t,T]$, $Y_s^{n,i,tx} = v^{i,n}(s, X_s^{tx})$. Using now inequality (3.6) we get:

$$Y_t^{n,i,tx} \le Y_t^{n+1,i,tx} \le E[\int_t^T \{\max_{i=1,m} |\psi_i(s, X_s^{tx})|\}ds]$$

since $Y_t^{n,i,tx}$ is deterministic. Therefore combining the polynomial growth of ψ_i and estimate (4.2) for X^{tx} we obtain:

$$v^{i,n}(t,x) \le v^{i,n+1}(t,x) \le C(1 + |x|^p)$$

for some constants C and p independent of n. In order to complete the proof it is enough now to set $v^i(t,x) := \lim_{n\to\infty} v^{i,n}(t,x), (t,x) \in [0,T] \times I\!R^k$ since $Y^{i,n,tx} \nearrow Y^{i,tx}$ as $n \to \infty$. $\Box$

We are now going to focus on the continuity of the functions $v^1, ..., v^m$. But first let us deal with some properties of the optimal strategy which exist thanks to Theorem 1.

Proposition 4 *Let $(\delta, u) = ((\tau_n)_{n\ge1}, (\xi_n)_{n\ge1})$ be an optimal strategy, then there exist two positive constant C and p which does not depend on t and x such that:*

$$\forall n \ge 1,\ P[\tau_n < T] \le \frac{C(1 + |x|^p)}{n}. \tag{4.7}$$

Proof: Recall the characterization of (3.8) that reads as:

$$Y_0^{1,tx} = sup_{(\delta,u)\in\mathcal{D}} E[\int_0^T \psi_{u_r}(r, X_r^{tx})dr - \sum_{k\geq 1} g_{u_{\tau_{k-1}} u_{\tau_k}}(\tau_k, X_{\tau_k}^{tx})1\!\!1_{[\tau_k<T]}].$$

Now if $(\delta, u) = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ is the optimal strategy then we have:

$$Y_0^{1,tx} = E[\int_0^T \psi_{u_r}(r, X_r^{tx})dr - \sum_{k\geq 1} g_{u_{\tau_{k-1}} u_{\tau_k}}(\tau_k, X_{\tau_k}^{tx})1\!\!1_{[\tau_k<T]}].$$

Taking into account that $g_{ij} \geq \alpha > 0$ for any $i \neq j$ we obtain:

$$E[\sum_{k=1,n} \alpha 1\!\!1_{[\tau_k<T]}] + Y_0^{1,tx} \quad \leq E[\int_0^T \psi_{u_r}(r, X_r^{tx})dr - \sum_{k\geq n+1} g_{u_{\tau_{k-1}} u_{\tau_k}}(\tau_k, X_{\tau_k}^{tx})1\!\!1_{[\tau_k<T]}].$$

But for any $k \leq n$, $[\tau_n < T] \subset [\tau_k < T]$ then:

$$\begin{aligned}\alpha n P[\tau_n < T] + Y_0^{1,tx} &\leq E[\int_0^T \psi_{u_r}(r, X_r^{tx})dr - \sum_{k\geq n+1} g_{u_{\tau_{k-1}} u_{\tau_k}}(\tau_k, X_{\tau_k}^{tx})1\!\!1_{[\tau_k<T]}] \\ &\leq E[\int_0^T \psi_{u_r}(r, X_r^{tx})dr].\end{aligned}$$

and then

$$\begin{aligned}n\alpha P[\tau_n < T] &\leq E[\int_0^T \mid \psi_{u_r}(r, X_r^{tx}) \mid dr] - Y_0^{1,tx} \\ &\leq E[\int_0^T \mid \psi_{u_r}(r, X_r^{tx}) \mid dr] - Y_0^{1,0,tx}.\end{aligned}$$

Finally taking into account the facts that ψ_i and $Y^{1,0,tx}$ are of polynomial growth and estimate (4.2) for X^{tx} to obtain the desired result. Note that the polynomial growth of $Y^{1,0,tx}$ stems from Proposition 3. □

Remark 2 *The estimate (4.7) is also valid for the optimal strategy if at the initial time the state of plant is an arbitrary $i \in \mathcal{I}$.* □

Next for $i \in \mathcal{I}$ let $(y_s^{i,tx}, z_s^{i,tx}, k_s^{i,tx})_{s\leq T}$ be the processes defined as follows:

$$\begin{cases} y^{i,tx}, k^{i,tx} \in \mathcal{S}^2 \text{ and } z^{i,tx} \in \mathcal{M}^{2,d};\ k^{i,tx} \text{ is non-decreasing and } k_0^{i,tx} = 0, \\ y_s^{i,tx} = \displaystyle\int_s^T \psi_i(r, X_r^{tx})1\!\!1_{[r\geq t]}dr - \int_s^T z_r^{i,tx}dB_r + k_T^{i,tx} - k_s^{i,tx},\ \ 0 \leq s \leq T,\ \ y_T^{i,tx} = 0 \\ y_s^{i,tx} \geq l_s^{i,tx} := \max\limits_{j\in\mathcal{I}^{-i}}\{-g_{ij}(t \vee s, X_{t\vee s}^{tx}) + y_s^{j,tx})\},\ \ \forall s \leq T, \\ \displaystyle\int_0^T (y_r^{i,tx} - l_r^{i,tx})dk_r^{i,tx} = 0. \end{cases} \tag{4.8}$$

The existence of $(y^{i,tx}, z^{i,tx}, k^{i,tx}), i \in \mathcal{I}$, is obtained in the same way as the one of $(Y^{i,tx}, Z^{i,tx}, K^{i,tx})$. On the other hand, thanks to uniqueness (see once more Remark 1), for any $(t, x) \in [0, T] \times I\!\!R^k$, for any $s \in [0, t]$ we have $y_s^{i,tx} = Y_t^{i,tx}$, $z_s^{i,tx} = 0$ and $k_s^{i,tx} = 0$.

We are now ready to give the main Theorem of this article.

Theorem 3 *The functions $(v^1, ..., v^m) : [0, T] \times I\!\!R^k \to I\!\!R$ are continuous and solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.4).*

Proof: First let us focus on continuity and let us show that v^1 is continuous. The same proof will be valid for the continuity of the other functions v^i $(i = 2, ..., m)$.

First the characterization (3.8) implies that:

$$y_0^{1,tx} = \sup_{(\delta,\xi)\in\mathcal{D}} E[\int_0^T \psi_{u_s}(s, X_s^{tx})1\!\!1_{[s\geq t]}ds - \sum_{n\geq 1} g_{u_{\tau_{n-1}}u_{\tau_n}}(\tau_n \vee t, X_{\tau_n\vee t}^{tx})1\!\!1_{[\tau_n<T]}]$$

On the other hand an optimal strategy (δ^*, ξ^*) exists and satisfies the estimates (4.7) with the same constants C and p. Next let $\epsilon > 0$ and $(t', x') \in B((t, x), \epsilon)$ and let us consider the following set of strategies:

$$\tilde{D} := \{(\delta, \xi) = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 0}) \in \mathcal{D} \text{ such that } \forall n \geq 1, P[\tau_n < T] \leq \frac{C(1 + (\epsilon + |x|)^p)}{n}\}.$$

Therefore the strategy (δ^*, ξ^*) belongs to $\tilde{D}$ and then we have:

$$\begin{aligned} y_0^{1,tx} &= \sup_{(\delta,\xi)\in\tilde{D}} E[\int_0^T \psi_{u_s}(s, X_s^{tx})1\!\!1_{[s\geq t]}ds - \textstyle\sum_{n\geq 1} g_{u_{\tau_{n-1}}u_{\tau_n}}(\tau_n \vee t, X_{\tau_n\vee t}^{tx})1\!\!1_{[\tau_n<T]}] \\ &= sup_{(\delta,u)\in\tilde{D}} E[\int_0^{\tau_n} \psi_{u_s}(s, X_s^{tx})1\!\!1_{[s\geq t]}ds \\ &\qquad - \textstyle\sum_{1\leq k\leq n} g_{u_{\tau_{k-1}}u_{\tau_k}}(t \vee \tau_k, X_{t\vee\tau_k}^{tx})1\!\!1_{[\tau_k<T]} + 1\!\!1_{[\tau_n<T]}y_{\tau_n}^{u_{\tau_n},tx}] \end{aligned}$$

and

$$\begin{aligned} y_0^{1,t'x'} &= \sup_{(\delta,\xi)\in\tilde{D}} E[\int_0^T \psi_{u_s}(s, X_s^{t'x'})1\!\!1_{[s\geq t']}ds - \textstyle\sum_{n\geq 1} g_{u_{\tau_{n-1}}u_{\tau_n}}(\tau_n \vee t', X_{\tau_n\vee t'}^{t'x'})1\!\!1_{[\tau_n<T]}] \\ &= sup_{(\delta,u)\in\tilde{D}} E[\int_0^{\tau_n} \psi_{u_s}(s, X_s^{t'x'})1\!\!1_{[s\geq t']}ds \\ &\qquad - \textstyle\sum_{1\leq k\leq n} g_{u_{\tau_{k-1}}u_{\tau_k}}(t' \vee \tau_k, X_{t'\vee\tau_k}^{t'x'})1\!\!1_{[\tau_k<T]} + 1\!\!1_{[\tau_n<T]}y_{\tau_n}^{u_{\tau_n},t'x'}]. \end{aligned}$$

The second equalities follow from the dynamical programming principle. It follows that:

$$\begin{aligned} y_0^{1,t'x'} - y_0^{1,tx} &\leq sup_{(\delta,u)\in\tilde{D}} E[\int_0^{\tau_n} \{\psi_{u_s}(s, X_s^{t'x'})1\!\!1_{[s\geq t']} - \psi_{u_s}(s, X_s^{tx})1\!\!1_{[s\geq t]}\}ds \\ &\quad - \textstyle\sum_{1\leq k\leq n}\{g_{u_{\tau_{k-1}}u_{\tau_k}}(t' \vee \tau_k, X_{t'\vee\tau_k}^{t'x'}) - g_{u_{\tau_{k-1}}u_{\tau_k}}(t \vee \tau_k, X_{t\vee\tau_k}^{tx})\}1\!\!1_{[\tau_k<T]} \\ &\quad + 1\!\!1_{[\tau_n<T]}\{y_{\tau_n}^{u_{\tau_n},t'x'} - y_{\tau_n}^{u_{\tau_n},tx}\}] \end{aligned} \tag{4.9}$$

Next w.l.o.g we assume that $t' < t$. Then from (4.9) we deduce that:

$$\begin{aligned} y_0^{1,t'x'} - y_0^{1,tx} &\leq sup_{(\delta,u)\in\tilde{D}} E[\int_0^{\tau_n} \{\psi_{u_s}(s, X_s^{t'x'}) - \psi_{u_s}(s, X_s^{tx}))1\!\!1_{[s\geq t]} + \psi_{u_s}(s, X_s^{t'x'})1\!\!1_{[t'\leq s<t]}\}ds \\ &\quad - \textstyle\sum_{1\leq k\leq n}\{g_{u_{\tau_{k-1}}u_{\tau_k}}(t' \vee \tau_k, X_{t'\vee\tau_k}^{t'x'}) - g_{u_{\tau_{k-1}}u_{\tau_k}}(t \vee \tau_k, X_{t\vee\tau_k}^{tx})\}1\!\!1_{[\tau_k<T]} \\ &\quad + 1\!\!1_{[\tau_n<T]}\{y_{\tau_n}^{u_{\tau_n},t'x'} - y_{\tau_n}^{u_{\tau_n},tx}\}] \\ &\leq E[\int_0^T \max_{j=1,m} |\psi_j(s, X_s^{t'x'}) - \psi_j(s, X_s^{tx}))|1\!\!1_{[s\geq t]} + \max_{j=1,m} |\psi_j(s, X_s^{t'x'})|1\!\!1_{[t'\leq s<t]}\}ds \\ &\quad + n \max_{i\neq j\in\mathcal{I}}\{\sup_{s\leq T} |g_{ij}(t' \vee s, X_{t'\vee s}^{t'x'}) - g_{ij}(t \vee s, X_{t\vee s}^{tx})|\} \\ &\quad + sup_{(\delta,u)\in\tilde{D}}(P[\tau_n < T])^{\frac{1}{2}}(2E[(y_{\tau_n}^{u_{\tau_n},t'x'})^2 + (y_{\tau_n}^{u_{\tau_n},tx})^2])^{\frac{1}{2}}. \end{aligned} \tag{4.10}$$

In the right-hand side of (4.10) the first term converges to 0 as $(t', x') \to (t, x)$. Next let us show that for any $i \neq j \in \mathcal{I}$,

$$E[\sup_{s\leq T} |g_{ij}(t' \vee s, X_{t'\vee s}^{t'x'}) - g_{ij}(t \vee s, X_{t\vee s}^{tx})|] \to 0 \text{ as } (t', x') \to (t, x).$$

Actually for any $\varrho > 0$ we have:

$$|g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})| \leq |g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{t'x'}_{t'\vee s})|1\!\!1_{[|X^{t'x'}_{t'\vee s}|\leq \varrho]} +$$
$$|g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{t'x'}_{t'\vee s})|1\!\!1_{[|X^{t'x'}_{t'\vee s}|\geq \varrho]} + |g_{ij}(t \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|.$$

Therefore we have:

$$\begin{aligned}
&E[\sup_{s\leq T} |g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|] \leq \\
&\quad E[\sup_{s\leq T}\{|g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{t'x'}_{t'\vee s})|1\!\!1_{[|X^{t'x'}_{t'\vee s}|\leq \varrho]}\}] + \\
&\quad E[\sup_{s\leq T}\{|g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{t'x'}_{t'\vee s})|\}1\!\!1_{[\sup_{s\leq T}|X^{t'x'}_{t'\vee s}|\geq \varrho]}] + \\
&\quad E[\sup_{s\leq T}\{|g_{ij}(t \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|\}1\!\!1_{[\sup_{s\leq T}|X^{t'x'}_{t'\vee s}|+\sup_{s\leq T}|X^{tx}_{t\vee s}|\geq \varrho]}] + \\
&\quad E[\sup_{s\leq T}\{|g_{ij}(t \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|\}1\!\!1_{[\sup_{s\leq T}|X^{t'x'}_{t'\vee s}|+\sup_{s\leq T}|X^{tx}_{t\vee s}|\leq \varrho]}]
\end{aligned}$$

But since g_{ij} is continuous then it is uniformly continuous on $[0,T] \times \{x \in I\!\!R^k, |x| \leq \varrho\}$. Henceforth for any $\epsilon_1 > 0$ there exists $\eta_{\epsilon_1} > 0$ such that for any $|t - t'| < \eta_{\epsilon_1}$ we have:

$$\sup_{s\leq T}\{|g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{t'x'}_{t'\vee s})|1\!\!1_{[|X^{t'x'}_{t'\vee s}|\leq \varrho]}\} \leq \epsilon_1. \tag{4.11}$$

Next using Cauchy-Schwarz's inequality and then Markov's one with the second term we obtain:

$$E[\sup_{s\leq T}\{|g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{t'x'}_{t'\vee s})|\}1\!\!1_{[\sup_{s\leq T}|X^{t'x'}_{t'\vee s}|\geq \varrho]}] \leq C(1 + |x'|^p)\varrho^{-\frac{1}{2}} \tag{4.12}$$

where C and p are real constants which are bound to the polynomial growth of g_{ij} and estimate (4.2). In the same way we have:

$$E[\sup_{s\leq T}\{|g_{ij}(t \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|\}1\!\!1_{[\sup_{s\leq T}|X^{t'x'}_{t'\vee s}|+\sup_{s\leq T}|X^{tx}_{t\vee s}|\geq \varrho]}] \leq C(1 + |x|^p + |x'|^p)\varrho^{-\frac{1}{2}} \tag{4.13}$$

Finally using the uniform continuity of g_{ij} on compact subsets, the continuity property (4.3) and the Lebesgue dominated convergence theorem to obtain that

$$E[\sup_{s\leq T}\{|g_{ij}(t \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|1\!\!1_{[\sup_{s\leq T}|X^{t'x'}_{t'\vee s}|+\sup_{s\leq T}|X^{tx}_{t\vee s}|\leq \varrho]}] \to 0 \text{ as } (t', x') \to (t, x). \tag{4.14}$$

Taking now into account (4.11)-(4.14) we have:

$$\limsup_{(t',x')\to(t,x)} E[\sup_{s\leq T} |g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|] \leq \epsilon_1 + C(1 + |x|^p)\varrho^{-\frac{1}{2}}.$$

As ϵ_1 and ϱ are arbitrary then making $\epsilon_1 \to 0$ and $\varrho \to +\infty$ to obtain that:

$$\lim_{(t',x')\to(t,x)} E[\sup_{s\leq T} |g_{ij}(t' \vee s, X^{t'x'}_{t'\vee s}) - g_{ij}(t \vee s, X^{tx}_{t\vee s})|] = 0.$$

Thus the claim is proved.

Finally let us focus on the last term in (4.10). Since $(\delta, u) \in \tilde{D}$ then:

$$\begin{aligned}
sup_{(\delta,u)\in\tilde{D}}(P[\tau_n < T])^{\frac{1}{2}}(2E[(y^{u_{\tau_n},t'x'}_{\tau_n})^2 + (y^{u_{\tau_n},tx}_{\tau_n})^2])^{\frac{1}{2}} &\leq n^{-\frac{1}{2}} \sup_{(\delta,u)\in\tilde{D}}(2E[(y^{u_{\tau_n},t'x'}_{\tau_n})^2 + (y^{u_{\tau_n},tx}_{\tau_n})^2])^{\frac{1}{2}} \\
&\leq Cn^{-\frac{1}{2}}(1 + |x|^p + |x'|^p)
\end{aligned}$$

where C and p are appropriate constants which come from the polynomial growth of ψ_i, $i \in \mathcal{I}$, estimate (4.2) for the process X^{tx} and inequality (3.6). Going back now to (4.10), taking the limit as $(t', x') \to (t, x)$ to obtain:

$$\limsup_{(t',x')\to(t,x)} y_0^{1,t'x'} \leq y_0^{1,tx} + Cn^{-\frac{1}{2}}(1 + 2|x|^p).$$

As n is arbitrary then putting $n \to +\infty$ to obtain:

$$\limsup_{(t',x')\to(t,x)} y_0^{1,t'x'} \leq y_0^{1,tx}.$$

It implies that:

$$\limsup_{(t',x')\to(t,x)} y_0^{1,t'x'} = Y_{t'}^{1,t'x'} = v^1(t', x') \leq y_0^{1,tx} = Y_t^{1,tx} = v^1(t, x).$$

Therefore v^1 is upper semi-continuous. But v^1 is also lower semi-continuous, therefore it is continuous. In the same way we can show that v^2,...,v^m are continuous. As they are of polynomial growth then taking into account Theorem 2 to obtain that $(v^1, \ldots, v^m)$ is a viscosity solution for the system of variational inequalities with inter-connected obstacles (2.4). □

5 Uniqueness of the solution of the system

We are going now to address the question of uniqueness of the viscosity solution of the system (2.4). We have the following:

Theorem 4 *The solution in viscosity sense of the system of variational inequalities with inter-connected obstacles (2.4) is unique in the space of continuous functions on $[0, T] \times R^k$ which satisfy a polynomial growth condition, i.e., in the space*

$$\begin{aligned}\mathcal{C} := \{\varphi : [0, T] \times I\!\!R^k \to I\!\!R,\ &\textit{continuous and for any}\\ &(t, x),\ |\varphi(t, x)| \leq C(1 + |x|^\gamma) \textit{ for some constants } C \textit{ and } \gamma\}.\end{aligned}$$

Proof. We will show by contradiction that if $u_1, ..., u_m$ and $w_1, ..., w_m$ are a subsolution and a supersolution respectively for (2.4) then for any $i = 1, ..., m$, $u_i \leq w_i$. Therefore if we have two solutions of (2.4) then they are obviously equal. Actually for some $R > 0$ suppose there exists $(\bar{t}, \bar{x}, \bar{i}) \in (0, T) \times B_R \times \mathcal{I}$ $(B_R := \{x \in I\!\!R^k; |x| < R\})$ such that:

$$u_{\bar{i}}(\bar{t}, \bar{x}) - w_{\bar{i}}(\bar{t}, \bar{x}) = \eta > 0. \tag{5.1}$$

Let us take θ, λ and $\beta \in (0, 1]$ small enough, so that the following holds:

$$\begin{cases} \beta T < \frac{\eta}{5} \\ 2\theta|\bar{x}|^{2\gamma+2} < \frac{\eta}{5} \\ -\lambda w_{\bar{i}}(\bar{t}, \bar{x}) < \frac{\eta}{5} \\ \frac{\lambda}{\bar{t}} < \frac{\eta}{5}. \end{cases} \tag{5.2}$$

Here γ is the growth exponent of the functions which w.l.o.g we assume integer and ≥ 2. Then, for a small $\epsilon > 0$, let us define:

$$\Phi_\epsilon^i(t,x,y) = u_i(t,x) - (1-\lambda)w_i(t,y) - \frac{1}{2\epsilon}|x-y|^{2\gamma} - \theta(|x|^{2\gamma+2} + |y|^{2\gamma+2}) + \beta t - \frac{\lambda}{t}. \tag{5.3}$$

By the growth assumption on u_i and w_i, there exists a $(t_0, x_0, y_0, i_0) \in (0,T] \times \overline{B}_R \times \overline{B}_R \times \mathcal{I}$, such that:

$$\Phi_\epsilon^{i_0}(t_0,x_0,y_0) = \max_{(t,x,y,i)} \Phi_\epsilon^i(t,x,y).$$

On the other hand, from $2\Phi_\epsilon^{i_0}(t_0,x_0,y_0) \geq \Phi_\epsilon^{i_0}(t_0,x_0,x_0) + \Phi_\epsilon^{i_0}(t_0,y_0,y_0)$, we have

$$\frac{1}{2\epsilon}|x_0-y_0|^{2\gamma} \leq (u_{i_0}(t_0,x_0) - u_{i_0}(t_0,y_0)) + (1-\lambda)(w_{i_0}(t_0,x_0) - w_{i_0}(t_0,y_0)), \tag{5.4}$$

and consequently $\frac{1}{2\epsilon}|x_0-y_0|^{2\gamma}$ is bounded, and as $\epsilon \to 0$, $|x_0-y_0| \to 0$. Since u_{i_0} and w_{i_0} are uniformly continuous on $[0,T] \times \overline{B}_R$, then $\frac{1}{2\epsilon}|x_0-y_0|^{2\gamma} \to 0$ as $\epsilon \to 0$.

Next let us show that $t_0 < T$. Actually if $t_0 = T$ then,

$$\Phi_\epsilon^{\overline{i}}(\overline{t},\overline{x},\overline{x}) \leq \Phi_\epsilon^{i_0}(T,x_0,y_0),$$

and,

$$u_{\overline{i}}(\overline{t},\overline{x}) - (1-\lambda)w_{\overline{i}}(\overline{t},\overline{x}) - 2\theta|\overline{x}|^{2\gamma+2} + \beta\overline{t} - \frac{\lambda}{\overline{t}} \leq \beta T - \frac{\lambda}{T},$$

since $u_{i_0}(T,x_0) = w_{i_0}(T,y_0) = 0$. Then thanks to (5.1) we have,

$$\begin{aligned} \eta &\leq -\lambda w_{\overline{i}}(\overline{t},\overline{x}) + \beta T + 2\theta|\overline{x}|^{2\gamma+2} + \tfrac{\lambda}{\overline{t}} \\ \eta &< \tfrac{4}{5}\eta. \end{aligned}$$

which yields a contradiction and we have $t_0 \in (0,T)$. We now claim that:

$$u_{i_0}(t_0,x_0) - \max_{j \in \mathcal{I}^{-i_0}} \{-g_{i_0 j}(t_0,x_0) + u_j(t_0,x_0)\} > 0. \tag{5.5}$$

Indeed if

$$u_{i_0}(t_0,x_0) - \max_{j \in \mathcal{I}^{-i_0}} \{-g_{i_0 j}(t_0,x_0) + u_j(t_0,x_0)\} \leq 0$$

then there exists $k \in \mathcal{I}^{-i_0}$ such that:

$$u_{i_0}(t_0,x_0) \leq -g_{i_0 k}(t_0,x_0) + u_k(t_0,x_0).$$

From the supersolution property of $w_{i_0}(t_0,y_0)$, we have

$$w_{i_0}(t_0,y_0) \geq \max_{j \in \mathcal{I}^{-i_0}} (-g_{i_0 j}(t_0,y_0) + w_j(t_0,y_0))$$

then

$$w_{i_0}(t_0,y_0) \geq -g_{i_0 k}(t_0,y_0) + w_k(t_0,y_0).$$

It follows that:

$$u_{i_0}(t_0,x_0)-(1-\lambda)w_{i_0}(t_0,y_0)-(u_k(t_0,x_0)-(1-\lambda)w_k(t_0,y_0))\leq(1-\lambda)g_{i_0k}(t_0,y_0)-g_{i_0k}(t_0,x_0).$$

Now since $g_{ij}\geq\alpha>0$, for every $i\neq j$, and taking into account of (5.3) to obtain:

$$\Phi_\epsilon^{i_0}(t_0,x_0,y_0)-\Phi_\epsilon^k(t_0,x_0,y_0)<-\alpha\lambda+g_{i_0k}(t_0,y_0)-g_{i_0k}(t_0,x_0).$$

But this contradicts the definition of i_0, since g_{i_0k} is uniformly continuous on $[0,T]\times\overline{B}_R$ and the claim (5.5) holds.

Next let us denote

$$\varphi_\epsilon(t,x,y)=\frac{1}{2\epsilon}|x-y|^{2\gamma}+\theta(|x|^{2\gamma+2}+|y|^{2\gamma+2})-\beta t+\frac{\lambda}{t}. \tag{5.6}$$

Then we have:

$$\begin{cases} D_t\varphi_\epsilon(t,x,y)=-\beta-\frac{\lambda}{t^2},\\ D_x\varphi_\epsilon(t,x,y)=\frac{\gamma}{\epsilon}(x-y)|x-y|^{2\gamma-2}+\theta(2\gamma+2)x|x|^{2\gamma},\\ D_y\varphi_\epsilon(t,x,y)=-\frac{\gamma}{\epsilon}(x-y)|x-y|^{2\gamma-2}+\theta(2\gamma+2)y|y|^{2\gamma},\\ \\ B(t,x,y)=D^2_{x,y}\varphi_\epsilon(t,x,y)=\frac{1}{\epsilon}\begin{pmatrix} a_1(x,y) & -a_1(x,y)\\ -a_1(x,y) & a_1(x,y)\end{pmatrix}+\begin{pmatrix} a_2(x) & 0\\ 0 & a_2(y)\end{pmatrix}\\ \\ \text{with } a_1(x,y)=\gamma|x-y|^{2\gamma-2}I+\gamma(2\gamma-2)(x-y)(x-y)^*|x-y|^{2\gamma-4} \text{ and}\\ a_2(x)=\theta(2\gamma+2)|x|^{2\gamma}I+2\theta\gamma(2\gamma+2)xx^*|x|^{2\gamma-2}.\end{cases} \tag{5.7}$$

Taking into account (5.5) then applying the result by Crandall et al. (Theorem 8.3, [6]) to the function

$$u_{i_0}(t,x)-(1-\lambda)w_{i_0}(t,y)-\varphi_\epsilon(t,x,y)$$

at the point (t_0,x_0,y_0), for any $\epsilon_1>0$, we can find $c,d\in I\!R$ and $X,Y\in S_k$, such that:

$$\begin{cases} (c,\frac{\gamma}{\epsilon}(x_0-y_0)|x_0-y_0|^{2\gamma-2}+\theta(2\gamma+2)x_0|x_0|^{2\gamma},X)\in J^{2,+}(u_{i_0}(t_0,x_0)),\\ (-d,\frac{\gamma}{\epsilon}(x_0-y_0)|x_0-y_0|^{2\gamma-2}-\theta(2\gamma+2)y_0|y_0|^{2\gamma},Y)\in J^{2,-}((1-\lambda)w_{i_0}(t_0,y_0)),\\ c+d=D_t\varphi_\epsilon(t_0,x_0,y_0)=-\beta-\frac{\lambda}{t_0^2} \text{ and finally}\\ -(\frac{1}{\epsilon_1}+||B(t_0,x_0,y_0)||)I\leq\begin{pmatrix} X & 0\\ 0 & -Y\end{pmatrix}\leq B(t_0,x_0,y_0)+\epsilon_1 B(t_0,x_0,y_0)^2.\end{cases} \tag{5.8}$$

Taking now into account (5.5), and the definition of viscosity solution, we get:

$$\begin{aligned} &-c-\tfrac{1}{2}Tr[\sigma^*(t_0,x_0)X\sigma(t_0,x_0)]-\langle\tfrac{\gamma}{\epsilon}(x_0-y_0)|x_0-y_0|^{2\gamma-2}+\\ &\qquad\qquad\theta(2\gamma+2)x_0|x_0|^{2\gamma},b(t_0,x_0)\rangle-\psi_{i_0}(t_0,x_0)\leq 0 \text{ and}\\ &d-\tfrac{1}{2}Tr[\sigma^*(t_0,y_0)Y\sigma(t_0,y_0)]-\langle\tfrac{\gamma}{\epsilon}(x_0-y_0)|x_0-y_0|^{2\gamma-2}-\\ &\qquad\qquad\theta(2\gamma+2)y_0|y_0|^{2\gamma},b(t_0,y_0)\rangle-(1-\lambda)\psi_{i_0}(t_0,y_0)\geq 0\end{aligned}$$

which implies that:

$$
\begin{aligned}
-c-d \quad &\le \tfrac{1}{2}Tr[\sigma^*(t_0,x_0)X\sigma(t_0,x_0)-\sigma^*(t_0,y_0)Y\sigma(t_0,y_0)] \\
&+\langle \tfrac{\gamma}{\epsilon}(x_0-y_0)|x_0-y_0|^{2\gamma-2}, b(t_0,x_0)-b(t_0,y_0)\rangle \\
&+\langle \theta(2\gamma+2)x_0|x_0|^{2\gamma}, b(t_0,x_0)\rangle+\langle \theta(2\gamma+2)y_0|y_0|^{2\gamma}, b(t_0,y_0)\rangle \\
&+\psi_i(t_0,x_0)-(1-\lambda)\psi_i(t_0,y_0).
\end{aligned}
\tag{5.9}
$$

But from (5.7) there exist two constants C and C_1 such that:

$$||a_1(x_0,y_0)|| \le C|x_0-y_0|^{2\gamma-2} \text{ and } (||a_2(x_0)|| \vee ||a_2(y_0)||) \le C_1\theta.$$

As

$$B=B(t_0,x_0,y_0)=\frac{1}{\epsilon}\begin{pmatrix} a_1(x_0,y_0) & -a_1(x_0,y_0) \\ -a_1(x_0,y_0) & a_1(x_0,y_0)\end{pmatrix}+\begin{pmatrix} a_2(x_0) & 0 \\ 0 & a_2(y_0)\end{pmatrix}$$

then

$$B \le \frac{C}{\epsilon}|x_0-y_0|^{2\gamma-2}\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}+C_1\theta I.$$

It follows that:

$$B+\epsilon_1B^2 \le C(\frac{1}{\epsilon}|x_0-y_0|^{2\gamma-2}+\frac{\epsilon_1}{\epsilon^2}|x_0-y_0|^{4\gamma-4})\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}+C_1\theta I \tag{5.10}$$

where C and C_1 which hereafter may change from line to line. Choosing now $\epsilon_1=\epsilon$, yields the relation

$$B+\epsilon_1B^2 \le \frac{C}{\epsilon}(|x_0-y_0|^{2\gamma-2}+|x_0-y_0|^{4\gamma-4})\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}+C_1\theta I. \tag{5.11}$$

Now, from (2.1), (5.8) and (5.11) we get:

$$\frac{1}{2}Tr[\sigma^*(t_0,x_0)X\sigma(t_0,x_0)-\sigma^*(t_0,y_0)Y\sigma(t_0,y_0)] \le \frac{C}{\epsilon}(|x_0-y_0|^{2\gamma}+|x_0-y_0|^{4\gamma-2})+C_1\theta(1+|x_0|^2+|y_0|^2).$$

Next

$$\langle \frac{\gamma}{\epsilon}(x_0-y_0)|x_0-y_0|^{2\gamma-2}, b(t_0,x_0)-b(t_0,y_0)\rangle \le \frac{C^2}{\epsilon}|x_0-y_0|^{2\gamma}$$

and finally,

$$\langle \theta(2\gamma+2)x_0|x_0|^{2\gamma}, b(t_0,x_0)\rangle+\langle \theta(2\gamma+2)y_0|y_0|^{2\gamma}, b(t_0,y_0)\rangle \le \theta C(1+|x_0|^{2\gamma+2}+|y_0|^{2\gamma+2}).$$

So that by plugging into (5.9) and note that $\lambda>0$ we obtain:

$$
\begin{gathered}
\beta \le \tfrac{C}{\epsilon}(|x_0-y_0|^{2\gamma}+|x_0-y_0|^{4\gamma-2})+C_1\theta(1+|x_0|^2+|y_0|^2)+\tfrac{C^2}{\epsilon}|x_0-y_0|^{2\gamma}+ \\
\theta C(1+|x_0|^{2\gamma+2}+|y_0|^{2\gamma+2})+\psi_{i_0}(t_0,x_0)-(1-\lambda)\psi_{i_0}(t_0,y_0).
\end{gathered}
$$

By sending $\epsilon\to 0$, $\lambda\to 0$, $\theta\to 0$ and taking into account of the continuity of ψ_{i_0} and $\gamma\ge 2$, we obtain $\beta\le 0$ which is a contradiction. The proof of Theorem 4 is now complete. □

As a by-product we have the following corollary:

Corollary 1 *Let $(v^1, ..., v^m)$ be a viscosity solution of (2.4) which satisfies a polynomial growth condition then for $i = 1, ..., m$ and $(t, x) \in [0, T] \times I\!R^k$,*

$$v^i(t,x) = \sup_{(\delta,\xi)\in\mathcal{D}_t^i} E[\int_t^T \psi_{u_s}(s, X_s^{tx})ds - \sum_{n\geq 1} g_{u_{\tau_{n-1}} u_{\tau_n}}(\tau_n, X_{\tau_n}^{tx})1\!\!1_{[\tau_n<T]}].$$

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